Minimax Programming in Complex Spaces : Necessary and Sufficient Optimality Conditions (Nonlinear Analysis and Convex Analysis)

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Minimax Programming in Complex Spaces[1,2]

—Necessary and Sufficient Optimality Conditions—

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Abstract.

In this note we study a nondifferentiable minimax programming in complex spaces. We establish the Kuhn-Tucker type necessary optimality conditions, and the existence theorem for optimality in complex programming under the framework of generalized convexity.

Key words: complex minimax programming, convex, pseudoconvex/quasiconvex in complex spaces, optimality.

1. Introduction

Mathematical programming in complex space was first studied by Levinson at 1966 for linear programming (LP). Shortly later Swarap and Sharma in 1970 studied for linear fractional programming (LFP). Henceafter nonlinear complex programming for fractional or nonfractional were treated by numerous authors. For instance Mond and Craven (1975), Das and Swarup (1977), Datta and Bhata (1984), and others, Hanson, Saxena, Jain, Ferrero, Lai, Liu and Schaible etc. were also studied complex programming for nonlinear fractional or nonfractional in different viewpoint.

Recently Chen-Lai-Schaible introduced a generalized Charnes-Cooper variable transformation to change fractional complex programming into nonfractional programming, and prove that the optimal solution of complex fractional programming can be reduced to an optimal solution of the equivalent nonfractional programming and vice versa.

In programming problem, the existence of optimal solution, continuity, convexity, and its

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various generalization are valuable in analysis as well as in the existence of optimal solution for programming problems under the considered framework.

Applications of complex programming (cf. Lai and Liu [5]) could be employed to electrical networks with complex variable $z \in \mathbb{C}^n$ to representing the currents or voltages for element of network. Various fields in electric engineering are employed. Like blind deconvolution, blind equalization, minimal entropy, maximal kurtosis, and optimal receiver etc. for example in a given statistical signal processing, one will maximize the equalizer output kurtosis as

$$K(z) = \frac{\left|E(|z|^4) - 2\left(E(|z|^2)\right)^2 - |E(z^2)|^2\right|}{E(|z|^2)^2}$$

where E stands for expectation, and $|z|^2 = z \cdot \overline{z}$.

In this note we establish the necessary and sufficient optimality conditions for a nondifferentiable minimax complex programming.

2. Nondifferentiable minimax complex programming

Consider a complex programming as the form:

$$(P) \qquad \mathrm{Min}_{\zeta \in X} \sup_{\eta \in Y} Re\left[f(\zeta, \eta) + (z^H A z)^{1/2}\right]$$
$$\text{subject to } X = \left\{\zeta = (z, \overline{z}) \in \mathbb{C}^{2n} \mid -h(\zeta) \in S\right\}$$

where (1) $Y = \{\eta = (w, \overline{w}) \mid w \in \mathbb{C}^m\}$ is a compact subset in $\mathbb{C}^{2n}$,

(2) $A \in \mathbb{C}^{n \times n}$ is a positive semidefinite Harmitian matrix,

(3) S is a polyhedral cone in $\mathbb{C}^p$,

(4) $f(\cdot,\cdot)$ is continuous, and for each $\eta \in Y$,

(5) $f(\cdot, \eta) : \mathbb{C}^{2n} \to \mathbb{C}$ and $h(\cdot) : \mathbb{C}^{2n} \to \mathbb{C}^p$ are analytic in $\zeta = (z, \overline{z}) \in Q \subset \mathbb{C}^{2n}$, $Q = \{(z, \overline{z}) \mid z \in \mathbb{C}^n\}$ is a linear manifold over real field.

Problem (P) is nondifferentiable programming if the optimal point $\zeta_0 = (z_0, \overline{z_0})$ with $z_0^H A z_0 = 0$, the term $z^H A z$ vanishes in a neighborhood of z_0. So $(z^H A z)^{1/2}$ is nondifferentiable at z_0, and problem (P) is then nondifferentiable.

Remark 2.1.

(a) If Y vanishes, problem (P) is reduced to:

$$(P_1) \qquad \text{Minimize } Re\left[f(\zeta) + (z^H A z)^{1/2}\right]$$
$$\text{subject to } \zeta = (z, \overline{z}) \in X = \{\zeta = (z, \overline{z}) \mid -h(\zeta) \in S\}$$

which is a nondifferentiable problem given in Mond and Craven [11].

(b) If $A = 0$, problem (P) becomes a differentiable complex programming:

(P_0) $\quad \mathrm{Min}_{\zeta\in X} \sup_{\eta\in Y} \ Ref(\zeta,\eta).$ s.t. $-h(\zeta) \in S$. (see Datta-Bhatia [4])

(c) Problem (P_0) extended the real minimax programming given by Schmittendorff [12]:

(P_r) $\quad \mathrm{Min}_{x\in X\subset\mathbb{R}^n} \sup_{y\in Y\subset\mathbb{R}^m} \ f(x,y)$ s.t. $h(x) \le 0$ in $\mathbb{R}^p$,

where $f(\cdot,\cdot)$ and $h(\cdot)$ are C^1 functions.

Remark 2.2. We give some complementary explanations, as follows:

(a) Ployhedral cone S in $\mathbb{C}^p$ means that there is a positive integer k and a matrix $B \in \mathbb{C}^{k\times p}$ such that $S = \{\xi \in \mathbb{C}^p \mid Re(B\xi) \ge 0\}$.

(b) The dual cone S^* of S is defined by the set $S^* = \{\mu \in \mathbb{C}^p \mid Re\langle\xi,\mu\rangle \ge 0 \text{ for } \xi \in S\}$. Obvious that $(S^*)^* = S$.

(c) For $s_0 \in S$, the set $S(s_0)$ is defined by the intersection of those closed half spaces including s_0 in their boundaries. Thus if $s_0 \in int(S)$, then $S(s_0) = \mathbb{C}^p$, the whole space.

3. Necessary optimality conditions

Definition 3.1. *The problem (P) is said to satisfy the* **constraint qualification** *at a point $\zeta_0 = (z_0, \overline{z_0})$, if for any nonzero $\mu \in S^* \subset \mathbb{C}^p$,*

$$\langle\ h'_\zeta(\zeta_0)(\zeta-\zeta_0), \mu\ \rangle \ne 0, \quad \text{for } \zeta \ne \zeta_0. \tag{3.1}$$

Lemma 3.1. *The constraint qualification (3.1) for problem (P) is equivalent to*

$$\mu^T\overline{\nabla_z h(\zeta_0)} + \mu^H \nabla_{\bar z} h(\zeta_0) = 0 \ \textit{only if}\ \mu = 0, \tag{3.2}$$

where $\mu^H = \overline{\mu^T}$.

Indeed, $\langle\ h'_\zeta(\zeta_0)(\zeta-\zeta_0), \mu\ \rangle$

$$= \Big\langle\ \nabla_z h(\zeta_0)(z-z_0) + \nabla_{\bar z} h(\zeta_0)(\ \overline{z-z_0}\ ), \mu\ \Big\rangle$$

$$= \overline{\mu}^T \nabla_z h(\zeta_0)(z-z_0) + \overline{\mu}^T \nabla_{\bar z} h(\zeta_0)(\overline{z-z_0})$$

$$= \Big\langle\ z-z_0\ ,\ \mu^T\overline{\nabla_z h(\zeta_0)}\ \Big\rangle + \Big\langle\ \mu^H \nabla_{\bar z} h(\zeta_0)\ ,\ z-z_0\ \Big\rangle.$$

So the real part of above identity (3.1) is equal to

$$Re\ \left[\langle z - z_0, \mu^T \overline{\nabla_z h(\zeta_0)} + \mu^H \nabla_{\bar{z}} h(\zeta_0)\ \rangle\right] \neq 0 \text{ if } \mu \neq 0 \text{ in } \mathbb{C}^p.$$

That is equivalently to the expression (3.2).

The necessary optimality condition follows easily from Kuhn-Tucker type conditions as the following:

Theorem 3.1. *Let $\zeta_0 = (z_0, \overline{z_0}) \in Q$ be (P_0)-optimal. Suppose that (P_0) satisfies the constraint qualification at ζ_0. Then there exist $0 \neq \mu \in S^* \subset \mathbb{C}^p$ and integer k with properties*

(i) $\eta_i \in Y(\zeta_0),\ i = 1, \cdots, k,$ *where*

$$Y(\zeta_0) = \{\eta \in Y \mid Re\ f(\zeta_0, \eta) = \sup_{\nu \in Y} Re\ f(\zeta_0, \nu)\},$$

(ii) multipliers $\lambda_i > 0,\ i = 1, \cdots, k,\ \sum_{i=1}^{k} \lambda_i = 1$

such that the Lagrangian $\varphi(\zeta) = \sum_{i=1}^{k} \lambda_i f(\zeta, \eta_i) + \langle h(\zeta), \mu \rangle$ satisfies the Kuhn-Tucker condition at ζ_0. That is,

$$\sum_{i=1}^{k} \lambda_i f'_\zeta(\zeta_0, \eta_i)(\zeta - \zeta_0) + \left\langle h'_\zeta(\zeta_0)(\zeta - \zeta_0), \mu \right\rangle = 0 \tag{3.3}$$

$$Re\ \left\langle h(\zeta_0), \mu \right\rangle = 0. \tag{3.4}$$

Proof. It follows from the compactness of Y in $\mathbb{C}^{2m}$ that there exist finite k points $\eta_1, \cdots, \eta_k \in Y(\zeta_0)$ satisfying conditions (i) and (ii), and hence the Lagrangian $\varphi(\zeta)$ satisfies the Kuhn-Tucker type conditions. $\square$

Remark 3.1. The real part of the left hand side of (3.3) deduces the real part of

$$\left\langle\ z - z_0\ ,\ \sum_{i=1}^{k} \lambda_i \left[\overline{\nabla_z f(\zeta_0, \eta_i)} + \nabla_{\bar{z}} f(\zeta_0, \eta_i)\right] + \left(\mu^T \overline{\nabla_z h(\zeta_0)} + \mu^H \nabla_{\bar{z}} h(\zeta_0)\right)\ \right\rangle$$

It follows that

$$\sum_{i=1}^{k} \lambda_i \left[\overline{\nabla_z f(\zeta_0, \eta_i)} + \nabla_{\bar{z}} f(\zeta_0, \eta_i)\right] + \mu^T \overline{\nabla_z h(\zeta_0)} + \mu^H \nabla_{\bar{z}} h(\zeta_0) = 0. \tag{3.5}$$

Mond [10] employed Eisenberg transformation theorem to establish the following

Lemma 3.2. *Let $E \in \mathbb{C}^{p \times n}$, $A \in \mathbb{C}^{n \times n}$ and $b \in \mathbb{C}^n$, $\mu \in S^* \subset \mathbb{C}^p$. Then the following two statements are equivalent*

(i) $E^H \mu = Au + b,\ u^H A u \leq 1$ *has solution* $u \in \mathbb{C}^n$.

(ii) If $Ez \in S \subset \mathbb{C}^p$ *for* $z \in \mathbb{C}^n$, *then* $Re\ \left[(z^H A z)^{1/2} + b^H z\right] \geq 0.$ $\square$

By this Lemma, Mond reduced the generalized Schwarz inequality in complex space:

$$Re(z^H Au) \leq (z^H Az)^{1/2}(u^H Au)^{1/2}, \tag{3.6}$$

The equality of (3.6) holds if $Az = \lambda Au$ or $z = \lambda\mu$ for $\lambda \geq 0$.

Accordingly Mond and Creven [11] proved the Kuhn-Tucker type necessary optimality conditions hold for problem (P) provided the optimal solution $\zeta_0 = (z_0, \overline{z_0}) \in Q$ satisfying $z_0^H Az_0 > 0$. That is

Theorem 3.2 *Let $\zeta_0 = (z_0, \overline{z_0}) \in Q$ be a (P)-optimal. Suppose that the constraint qualification holds for (P) at ζ_0 and $z_0^H Az_0 = \langle Az_0, z_0\rangle > 0$. Then there exist $0 \neq \mu \in S^* \subset \mathbb{C}^p$, $u \in \mathbb{C}^n$ and integer k with*

(i) finite points $\eta_i \in Y(\zeta_0)$, $i = 1, \cdots, k$;

(ii) multipliers $\lambda_i > 0$, $i = 1, \cdots, k$, $\sum_{i=1}^k \lambda_i = 1$

such that $\sum_{i=1}^{k} \lambda_i f(\zeta, \eta_i) + \langle \mu, h(\zeta)\rangle + \langle Az, z\rangle^{1/2}$ *satisfies the following conditions*

$$\sum_{i=1}^{k} \lambda_i \Big[\overline{\nabla_z f(\zeta_0, \eta_i)} + \nabla_{\bar{z}} f(\zeta_0, \eta_i) + Au\Big] + \Big(\mu^T \overline{\nabla_z h(\zeta_0)} + \mu^H \nabla_{\bar{z}} h(\zeta_0)\Big) = 0; \tag{3.7}$$

$$Re\langle\, h(\zeta_0), \mu\,\rangle = 0; \tag{3.8}$$

$$u^H Au \leq 1; \tag{3.9}$$

$$(z_0^H Az_0)^{1/2} = Re(z_0^H Au). \tag{3.10}$$

Proof. Since A is a positive definite Harmitian matrix and for each $\eta \in Y$, $f(\zeta, \eta)$ is analytic in ζ, thus for nonzero $\mu \in S^* \subset \mathbb{C}^p$, the function $f(\zeta, \eta) + (z^H Az)^{1/2} + \langle \mu, h(\zeta)\rangle$ is analytic at ζ_0. Hence by Theorem 3.1, there exist k, $\eta_i \in Y(\zeta_0)$, $\lambda_i > 0$, $i = 1, \cdots, k$ and $\sum_{i=1}^k \lambda_i = 1$ in conditions (i), (ii) such that

$$\sum_{i=1}^{k} \lambda_i \Big[\overline{\nabla_z f(\zeta_0, \eta_i)} + \nabla_{\bar{z}} f(\zeta_0, \eta_i)\Big] + \Big(\mu^T \overline{\nabla_z h(\zeta_0)} + \mu^H \nabla_{\bar{z}} h(\zeta_0)\Big) + \frac{Az_0}{\langle\, Az_0, z_0\,\rangle^{1/2}} = 0$$

and $Re\langle\, \mu, h(\zeta_0)\,\rangle = 0$. Putting $u = z_0/\langle\, Az_0, z_0\,\rangle^{1/2}$, it follows that (3.7)~(3.10) hold. □

In Theorem 3.2, if the (P)-optimal $\zeta_0 = (z_0, \overline{z_0})$ satisfies $\langle Az_0, z_0\rangle = 0$, then the objective of (P) is not analytic at z_0 (or ζ_0). The result of Theorem 3.2 still hold. The further assumption needs that a set $Z_{\tilde{\eta}(\zeta_0)}$ defined later will be empty. Since $Y(\zeta_0) \subset Y$ is compact, there is an integer $k > 0$ with $\eta_i \in Y(\zeta_0)$, $\lambda_i > 0$, $i = 1, \cdots, k$, $\sum_{i=1}^k \lambda_i = 1$ satisfying (i) and (ii). Let $\tilde{\eta} = (\eta_1, \cdots, \eta_k) \in Y(\zeta_0)^k$. If $\langle Az_0, z_0 \rangle = 0$ for $\zeta_0 = (z_0, \overline{z_0})$, we define

$$\begin{aligned} Z_{\tilde{\eta}}(\zeta_0) = \Big\{ \zeta \in \mathbb{C}^{2n} \;\Big|\; & -h'_\zeta(\zeta_0)\zeta \in S(-h(\zeta_0)), \\ & \zeta = (z, \overline{z}) \in Q \text{ and} \\ & Re\Big[\sum_{i=1}^k \lambda_i f'_\zeta(\zeta_0, \eta_i)\zeta + \langle Az, z \rangle^{1/2} \Big] < 0 \Big\}. \end{aligned}$$

Then we can prove the necessary theorem as following.

Theorem 3.3. *Let $\zeta_0 = (z_0, \overline{z_0}) \in Q$ be (P)-optimal. Suppose that problem (P) possess the constraint qualification at ζ_0, $\langle Az_0, z_0 \rangle = 0$ and $Z_{\tilde{\eta}}(\zeta_0) = \emptyset$. Then there exist a nonzero $\mu \in S^* \subset \mathbb{C}^p$ and a vector $u \in \mathbb{C}^n$ such that conditions (3.7)$\sim$(3.10) hold.*

4. Sufficient optimality conditions

A sufficient optimality theorem may be regarded as the inverse of necessary theorem with extra assumptions. We need several generalization for convexity of complex functions. Since a nonlinear analytic function have a convex real part, it must be considered that the complex functions are defined in the linear manifold $Q = \{\zeta = (z, \overline{z}) \in \mathbb{C}^{2n} \mid z \in \mathbb{C}^n\}$. For detail, one can consult Lai and Liu [5] and the references therein.

For each $\eta \in Y \subset \mathbb{C}^{2m}$, consider function $f(\cdot, \eta) : \mathbb{C}^{2m} \to \mathbb{C}$ and mapping $h(\cdot) : \mathbb{C}^{2n} \to \mathbb{C}^p$ that are analytic at $\zeta_0 = (z_0, \overline{z_0}) \in Q$. For any $\zeta \in Q$, we denote

$$I_1 = Re[f(\zeta, \eta) - f(\zeta_0, \eta)], \qquad J_1 = Re[f'_\zeta(\zeta_0)(\zeta - \zeta_0)];$$

and for $\mu \in S^* \subset \mathbb{C}^p$, denote

$$I_2 = Re\langle\, h(\zeta) - h(\zeta_0), \mu \,\rangle, \qquad J_2 = Re\langle\, h'_\zeta(\zeta_0)(\zeta - \zeta_0), \mu \,\rangle.$$

Then the generalized convexities are defined as following.

Definition 4.1 *The real part of analytic function $f(\cdot, \eta) : \mathbb{C}^{2n} \to \mathbb{C}$ is called, respectively,*

(i) **convex** *at $\zeta = \zeta_0$, if $I_1 \geq J_1$;*

(ii) **pseudoconvex (strictly)** *at $\zeta = \zeta_0$, if $J_1 \geq 0 \Rightarrow I_1 \geq 0$ ($I_1 > 0$);*

(iii) **quasiconvex** *at $\zeta = \zeta_0$, if $I_1 \leq 0 \Rightarrow J_1 \leq 0$.*

Definition 4.2. *The analytic mapping $h(\cdot) : \mathbb{C}^{2n} \to \mathbb{C}^p$ is called, respectively,*

(i) **convex** *at* $\zeta = \zeta_0$ *w.r.t. the polyhedral cone* S *in* $\mathbb{C}^p$,
if there exists $\mu \in S^* \subset \mathbb{C}^p$ *such that* $I_2 \geq J_2$;

(ii) **pseudoconvex (strictly)** *at* $\zeta = \zeta_0$ *w.r.t. to* S *in* $\mathbb{C}^p$,
if there exists $\mu \in S^* \subset \mathbb{C}^p$ *such that* $J_2 \geq 0 \Rightarrow I_2 \geq 0$ $(I_2 > 0)$;

(iii) **quasiconvex** *at* $\zeta = \zeta_0$ *w.r.t. to* S *in* $\mathbb{C}^p$,
if there exists $\mu \in S^* \subset \mathbb{C}^p$ *such that* $I_2 \leq 0 \Rightarrow J_2 \leq 0$.

Now we can state here three sufficient optimality theorems for a feasible solution of (P) becomes optimal.

Theorem 4.1. (Sufficient optimality conditions).

Let $\zeta_0 = (z_0, \overline{z_0}) \in Q$ *be a feasible solution of (P). Suppose that there exist* $\lambda_i > 0$ *with* $\sum_{i=1}^k \lambda_i = 1$, $\eta_i \in Y$, $i = 1, \cdots, k$, *and* $0 \neq \mu \in S^* \subset \mathbb{C}^p$, $u \in \mathbb{C}^n$ *satisfying conditions (3.7)∼(3.10) in Theorem 3.2. Further assume that any one of the following conditions (i), (ii) and (iii) holds:*

(i) $Re\Big[\sum_{i=1}^k \lambda_i f(\zeta, \eta_i) + z^H Au\Big]$ *is pseudoconvex on* $\zeta = (z, \overline{z}) \in Q$, $h(\zeta)$ *is quasiconvex on* Q *w.r.t.* $S \subset \mathbb{C}^p$;

(ii) $Re\Big[\sum_{i=1}^k \lambda_i f(\zeta, \eta_i) + z^H Au\Big]$ *is quasiconvex on* $\zeta = (z, \overline{z}) \in Q$ *and* $h(\zeta)$ *is strictly pseudoconvex on* Q *w.r.t.* $S \subset \mathbb{C}^p$;

(iii) $Re\Big[\sum_{i=1}^k \lambda_i f(\zeta, \eta_i) + z^H Au + \langle h(\zeta), \mu\rangle\Big]$ *is pseudoconvex on* $\zeta = (z, \overline{z}) \in Q$.

Then $\zeta_0 = (z_0, \overline{z_0})$ *is an optimal solution of (P).*

5. Further Plausible Work

As we have established (necessary and sufficient) optimality conditions, it is naturely arise a plausible problem that one may consider some duality models for the complex programming problem (P). We would like left it for later oportunity in details.

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